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THEORETICAL ANALYSIS AND DETERMINATION OF HYDRODYNAMIC AND THERMAL BOUNDARY LAYERS THROUGH OF GENERALIZED INTEGRAL TRANSFORM TECHNIQUE

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Abstract. *The present work has as object of study, development of the temperature and velocity profile and the determination of the thermal and hydrodynamic boundary layers thicknesses in the laminar flow. The equations of the momentum and energy are transformed using change of variable to determine of the hydrodynamic and thermal boundary layers thicknesses, respectively. The main objective of this article is to have a general view of the behavior of the thermal and hydrodynamic boundary layers in a conduction-external convection conjugated problem. To solve the coupled problem it was necessary to use, in the hydrodynamic field, the exact analytical solution of the hydrodynamic boundary layer thickness, while in the thermal field the relation between the Prandtl number and the thermal boundary layer thickness was used. GITT was able to solve the problem of the hydrodynamic and thermal boundary layers, making it possible to determine the hydrodynamic boundary layer thickness without using the similarity method, which shows an unprecedented treatment in solving this problem. The evolution of the position of the thermal and hydrodynamic boundary layer was calculated and compared with the Blasius solution and with approximate polynomial solutions, obtaining excellent results.*

Keywords: *Boundary Layer Thickness, Conduction-Convection Conjugate Problem, Generalized Integral Transform Technique*

1. INTRODUCTION

Boundary layer theory is considered the cornerstone of our knowledge of fluid flow over a surface that not only presents some intriguing physical phenomena of fluid dynamics, but is also fundamental to practical engineering problems. In 1908, H. Blasius published a paper discussing two-dimensional flow on a flat plate. The Blasius-derived boundary-layer equations were much simpler than the Navier-Stokes equations. Blasius found that these boundary-layer equations in certain cases can be reduced to a single ordinary differential equation for a solution of similarity, called the Blasius equation.

Until 1960, several researchers contributed significantly to solving the boundary layer problem using similarity solutions (Blasius, 1908; Millikan, 1936; Hartree, 1939; Goldstein, 1948; Terrill, 1960). With the evolution of the latest generation computers and the increase in the capacity of processors available on the market, has increased the number of solutions to equations that were previously considered impossible to solve due to the need for memory to run the programs, with this, several researchers have sought new solutions for Blasius problem using algorithms or solving the problem through series in order to find a solution that better represents the mathematical model and the physical phenomenon (Asaithambi, 2004; Fazio, 2009, 2013; Parveen, 2016; Jaguaribe, 2020).

The vast majority of heat transfer problems found in nature take into account the effects of heat conduction and convection, however, there are still great difficulties in simulating these problems with good precision and with reduced computational cost. Perelman (1961) used the term conjugated heat transfer to describe the coupled problem of convection heat transfer in the thermal boundary layer of a flow over a finite-thickness flat plate and the two-dimensional heat conduction in the solid wall. Identified a parameter that combined the fluid and solid conductivity ratio with Prandtl and Reynolds number. For this conjugated conduction-external convection problem, approximate solutions was found in the open literature available, using the Integral Method in conjunction with the Finite Difference Method or purely numerical solutions. It is worth mentioning that these works demonstrated the mathematical and / or computational difficulties associated with the solution of the conjugated problem(Sunden, 1989; Lachi *et al.*, 1996; Mosaad, 1999; Chida, 2000). Naveira *et al.* (2007) studied the conjugated transient external conduction-convection problem by applying a heat flow the wall through the Generalized Integral Transform Technique, using the Blasius solution in the hydrodynamic boundary layer.

The main objective of this article is to have a general view of the behavior of the hydrodynamic and thermal boundary layer in a conjugated conduction-external convection problem in steady state using the velocity profile under development through the application of the Generalized Integral Transform Technique analyzing the velocity and temperature field, using heat transfer coefficients defined a prior that were adopted in previous works in the literature with good results achieved.

2. PROBLEM FORMULATION

The considered problem involves laminar incompressible flow of a Newtonian fluid over a flat plate, with steady-state flow. The fluid flows with a free stream velocity u_∞ , which arrives at the plate front edge at the temperature T_∞ , conform show in Fig. 1. The wall is considered to participate on the heat transfer problem, with thickness, e, length, L, and related thermophysical properties. The boundary layer equations are assumed to be valid for the flow and heat transfer problem within the fluid, and the conjugated conduction–external convection problem is written as:

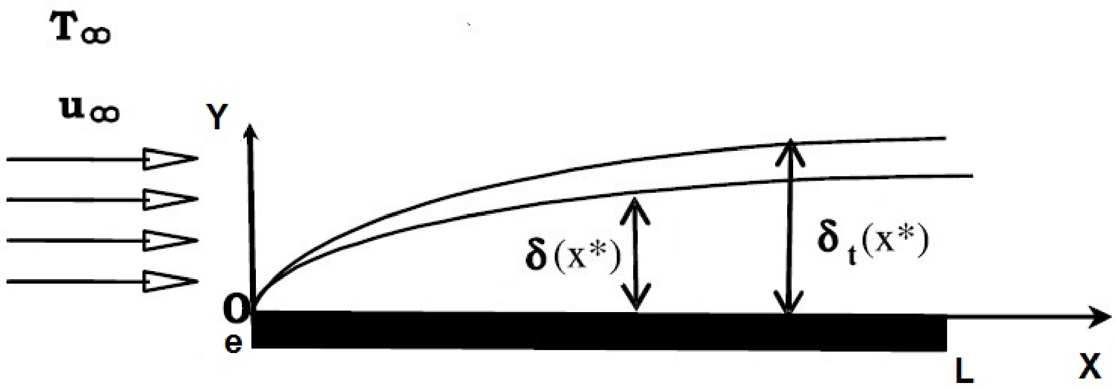

Figure 1. Description of physical problem and coordinates system.

Continuity:

$$\frac{\partial u(x^*,y^*)}{\partial x^*}+\frac{\partial v(x^*,y^*)}{\partial y^*} \;\; ; \;\; 0<x^*<L \;\; ; \;\; 0<y^*<\delta^*(x^*), \tag{1}$$

Momentum in x-direction:

$$u\frac{\partial u}{\partial x^*}+v\frac{\partial u}{\partial y^*}=-\frac{1}{\rho}\frac{dp}{dx^*}+\nu\frac{\partial^2 u}{\partial {y^*}^2} \;\; ; \;\; 0<x^*<L \;\; ; \;\; 0<y^*<\delta^*(x^*) \tag{2}$$

Momentum in y-direction:

$$\frac{\partial p}{\partial y^*}=0 \;\; ; \;\; 0<x^*<L \;\; ; \;\; 0<y^*<\delta^*(x^*) \tag{3}$$

Energy:

$$u\frac{\partial T(x^*,y^*)}{\partial x^*}+v\frac{\partial T(x^*,y^*)}{\partial y^*}=\alpha\frac{\partial^2 T(x^*,y^*)}{\partial {y^*}^2} \;\; ; \;\; 0<x^*<L \;\; ; \;\; 0<y^*<\delta^*(x^*) \tag{4}$$

with initial conditions

$$u(0,y^*) = U_\infty \ ; \ v(0,y^*) = 0 \ ; \ T(0,y^*) = T_\infty \ ; \ 0 < y^* < \infty \tag{5}$$

and boundary conditions

$$u(x^*,0) = 0 \ ; \ u(x^*,\delta^*(x^*)) = U_\infty \ ; \ v(x^*,0) = v(x^*,\delta^*(x^*)) = 0 \ ; \ \left.\frac{\partial T(x^*,y^*)}{\partial y^*}\right|_{y^*=0} = -\delta_t^*(x^*) \ ; \ T(x^*,0) = T_\infty \tag{6}$$

The hydrodynamic and conjugated conduction-convection problem can also be rewritten after introducing the following dimensionless variables:

$$U = \frac{u}{U_\infty} \ ; \ V = \frac{v}{U_\infty} \ ; \ x = \frac{x^*}{L} \ ; \ y = \frac{y^*}{L} \ ; \ \delta = \frac{\delta^*}{L} \ ; \ \delta_t = \frac{\delta_t^*}{L} \tag{7}$$

$$p^* = \frac{p}{LU_\infty^2} \ ; \ Re_L = \frac{U_\infty L}{\nu} \ ; \ Pe = \frac{U_\infty L}{\alpha} \ ; \ \theta = \frac{(T - T_\infty)}{\frac{q_{ref} L}{K}} \tag{8}$$

and introduce a domain regularization transformation for the spatial domain written as:

$$\eta = \frac{y}{\delta(x)} \ ; \ \eta_t = \frac{y}{\delta_t(x)} \ ; \ \chi = x \tag{9}$$

Then, the dimensionless form for the flow and energy equation after the domain transformation for the spatial domain written as:

Continuity:

$$\frac{\partial U}{\partial \chi} - \frac{1}{\delta(\chi)}\frac{d\delta(\chi)}{d\chi}\eta\frac{\partial U}{\partial \eta} + \frac{1}{\delta(\chi)}\frac{\partial V}{\partial \eta} = 0 \ ; \ 0 < \chi < 1 \ ; \ 0 < \eta < 1 \tag{10}$$

Momentum in χ-direction:

$$U\frac{\partial U}{\partial \chi} - \frac{U}{\delta(\chi)}\frac{d\delta(\chi)}{d\chi}\eta\frac{\partial U}{\partial \eta} + \frac{V}{\delta(\chi)}\frac{\partial U}{\partial \eta} = -\frac{dp^*}{d\chi} + \frac{1}{Re_L}\frac{\partial^2 U}{\partial \eta^2} \ ; \ 0 < \chi < 1 \ ; \ 0 < \eta < 1 \tag{11}$$

Momentum in η-direction:

$$-\frac{dp^*}{d\eta} = 0 \ ; \ 0 < \chi < 1 \ ; \ 0 < \eta < 1 \tag{12}$$

Energy:

$$U\left(\frac{\partial \theta}{\partial \chi} - \frac{\eta}{\delta_t(\chi)}\frac{d\delta_t(\chi)}{d\chi}\eta\frac{\partial \theta}{\partial \eta}\right) + \frac{V}{\delta_t(\chi)}\frac{\partial \theta}{\partial \eta} = \frac{1}{Pe}\frac{1}{\delta_t(\chi)^2}\frac{\partial^2 \theta}{\partial \eta^2} = 0 \ ; \ 0 < \chi < 1 \ ; \ 0 < \eta < 1 \tag{13}$$

with initial conditions

$$u(0,\eta) = 1 \ ; \ v(0,\eta) = 0 \ ; \ \theta(0,\eta) = 0 \tag{14}$$

and boundary conditions

$$U(\chi,0) = 0 \ ; \ U(\chi,1) = 1 \ ; \ V(\chi,0) = V(\chi,1) = 0 \ ; \ \frac{\partial \theta(\chi,\eta_t)}{\partial \eta_t}|_{\eta_t=0} = -\delta_t^*(x^*) \ ; \ 0 < \chi < L \tag{15}$$

3. SOLUTION METHODOLOGY

For the flow and thermal problem solution, since there is preferential convective direction aligned with the flow, the integral transformation was chosen to be operator solely in the transversal direction, along which diffusion predominates. As the proposed system of equations has non-homogeneous boundary conditions, in order to improving the performance of GITT, it is necessary to homogenize them. Filters and their solutions are written as:

$$U(\chi,\eta) = U^*(\chi,\eta) + UF(\eta) \;\; ; \;\; UF(\eta) = \eta \tag{16}$$

$$\theta(\chi,\eta_t) = \theta^*(\chi,\eta_t) + TF(\chi,\eta_t) \;\; ; \;\; TF(\chi,\eta_t) = \frac{\delta_t(\chi)}{2}(1-\eta_t)^2 \tag{17}$$

Applying the proposed filtering solutions, the continuity, momentum and energy equations is given by:

$$\frac{\partial U^*}{\partial \chi} - \frac{1}{\delta(\chi)}\frac{d\delta(\chi)}{d\chi}\eta\left(\frac{\partial U^*}{\partial \eta}+1\right) + \frac{1}{\delta(\chi)}\frac{\partial V}{\partial \eta} = 0 \;\; ; \;\; 0<\chi<1 \;\; ; \;\; 0<\eta<1, \tag{18}$$

$$(U^*+\eta)\left(\frac{\partial U^*}{\partial \chi} - \frac{1}{\delta(\chi)}\frac{d\delta(\chi)}{d\chi}\eta\left(\frac{\partial U^*}{\partial \eta}+1\right)\right) + \frac{V}{\delta(\chi)}\left(\frac{\partial U^*}{\partial \eta}+1\right) = -\frac{dp^*}{d\chi} + \frac{1}{Re_L}\frac{\partial^2 U^*}{\partial \eta^2} \tag{19}$$

$$(U^*+\eta)\left(\frac{\partial \theta^*}{\partial \chi} - \frac{\eta}{\delta_t(\chi)}\frac{d\delta_t(\chi)}{d\chi}\eta\frac{\partial \theta^*}{\partial \eta}\right) + \frac{V}{\delta_t(\chi)}\left(\frac{\partial \theta^*}{\partial \eta} - \frac{\partial TF}{\partial \eta}\right) = \frac{1}{Pe}\frac{1}{\delta_t(\chi)^2}\left(\frac{\partial^2 \theta^*}{\partial \eta^2} - \frac{\partial^2 TF}{\partial \eta^2}\right) = 0 \tag{20}$$

Following the basic steps of GITT (Cotta, 2020), the appropriate auxiliary problems for the process of integral transform, are given as follows:

Hydrodynamic Field

$$\frac{d^2\psi(\eta)}{d\eta^2} + \mu_i^2\psi(\eta) = 0 \;\; ; \;\; \psi(0)=0 \;\; ; \;\; \psi(1)=0, \tag{21}$$

which is readily solved eigenfunctions, normalized eigenfunctions, eigenvalues and norms, respectively, as:

$$\psi(\eta) = Sin[\mu_i(\eta)] \;\; ; \;\; \tilde{\psi}(\eta) = \frac{\psi(\eta)}{\sqrt{N_i}} \;\; ; \;\; \mu_i = i\pi \;\; ; \;\; N_i = \int_0^1 \psi_i(\eta)\,\psi_i(\eta)\,d\eta = \frac{1}{2} \;\; ; \;\; i=1,2,3... \tag{22}$$

Thermal Field

$$\frac{d^2\phi(\eta_t)}{d\eta_t^2} + \Omega_i^2\phi(\eta_t) = 0 \;\; ; \;\; \left.\frac{d\phi(\eta_t)}{d\eta_t^2}\right|_{\eta_t=0} = 0 \;\; ; \;\; \phi(1)=0, \tag{23}$$

which is readily solved eigenfunctions, normalized eigenfunctions, eigenvalues and norms, respectively, as:

$$\phi(\eta_t) = Cos[\Omega_i\eta_t] \;\; ; \;\; \tilde{\phi}(\eta_t) = \frac{\phi(\eta_t)}{\sqrt{M_i}} \;\; ; \;\; \Omega_i = \frac{(2i-1)\pi}{2} \;\; ; \;\; M_i = \int_0^1 \phi_i(\eta_t)\,\phi_i(\eta_t)\,d\eta_t = \frac{1}{2} \;\; ; \;\; i=1,2,3... \tag{24}$$

The eigenvalue problems Eqs. (21) and (23) allows definition of the following transform-inverse pairs:

$$\overline{U_j^*}(\chi) = \int_0^1 \tilde{\psi}_j(\eta)\,U^*(\chi,\eta)d\eta \;\; \rightarrow \;\; Transform \;\; ; \quad U^*(\chi,\eta) = \sum_{j=1}^{\infty}\tilde{\psi}_j(\eta)\,\overline{U_j}(\chi) \;\; \rightarrow \;\; Inverse \tag{25}$$

$$\overline{\theta_j^*}(\chi) = \int_0^1 \tilde{\phi}(\Omega_j)\theta^*(\chi,\eta_t)d\eta_t \;\; \rightarrow \;\; Transform \;\; ; \quad \theta^*(\chi,\eta_t) = \sum_{j=1}^{\infty}\tilde{\phi}_j(\eta_t)\,\overline{\theta_j}(\chi) \;\; \rightarrow \;\; Inverse \tag{26}$$

Applying the operators over Eqs. (19) and (20), followed by the inverse formula then results:
Momentum Equation:

$$\delta(\chi)^2\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}AV_{ijk}U_j(\chi)+\delta_{jk}BV_{ik}\right)\frac{\overline{dU_k}(\chi)}{d\chi}-\delta(\chi)\frac{d\delta(\chi)}{d\chi}\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}CV_{ijk}\overline{U_j}(\chi)+DV_{ik}+EV_{ik}\right)\overline{U_k}(\chi)$$
$$-\delta(\chi)\frac{d\delta(\chi)}{d\chi}FV_i=-\delta(\chi)^2\frac{dp}{d\chi}(\chi)FF_i-\frac{\mu_i^2}{Re_L}\overline{U_i}(\chi)\quad(27)$$

With the coefficients written below:

$$AV_{ijk}=\int_0^1\tilde{\psi}_i(\eta)\,\tilde{\psi}_j(\eta)\,\tilde{\psi}_k(\eta)\,d\eta+\int_0^1\tilde{\psi}_i(\eta)\,H_k(\eta)\frac{d\tilde{\psi}_j(\eta)}{d\eta}d\eta\ ;\ BV_{ij}=\int_0^1\tilde{\psi}_i(\eta)\,\eta\ \tilde{\psi}_j(\eta)\,d\eta+\int_0^1\tilde{\psi}_i(\eta)\,H_j(\eta)d\eta\quad(28)$$

$$CV_{ijk}=\int_0^1\tilde{\psi}_i(\eta)\,\eta\tilde{\psi}_j(\eta)\,\frac{d\tilde{\psi}_k(\eta)}{d\eta}d\eta+\int_0^1\tilde{\psi}_i(\eta)\,F_k(\eta)\,\frac{d\tilde{\psi}_j(\eta)}{d\eta}d\eta\ ;\ F_i(\eta)=\int_\eta^1\eta\frac{d\tilde{\psi}_i(\eta)}{d\eta}d\eta\quad(29)$$

$$DV_{ik}=\int_0^1\tilde{\psi}_i(\eta)\,\eta^2\frac{d\tilde{\psi}_k(\eta)}{d\eta}d\eta+\int_0^1\tilde{\psi}_i(\eta)\,\eta\tilde{\psi}_k(\eta)\,d\eta\ ;\ G(\eta)=\frac{1}{2}-\frac{\eta^2}{2}\ ;\ H_i(\eta)=\int_\eta^1\tilde{\psi}_i(\eta)\,d\eta\quad(30)$$

$$EV_{ik}=\int_0^1\tilde{\psi}_i(\eta)\,G(\eta)\frac{d\tilde{\psi}_k(\eta)}{d\eta}d\eta+\int_0^1\tilde{\psi}_i(\eta)\,H_k(\eta)d\eta\ ;\ FV_i=\int_0^1\tilde{\psi}_i(\eta)\,\eta^2d\eta+\int_0^1\tilde{\psi}_i(\eta)\,G(\eta)d\eta\quad(31)$$

Pressure gradient:

$$\frac{dp(\chi)}{d\chi}=2\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}AP_{jk}U_j(\chi)+BP_k\right)\frac{\overline{dU_k}(\chi)}{d\chi}-\frac{1}{Re_L\delta(\chi)^2}\sum_{j=1}^{\infty}\frac{(Cos[\mu_j]\mu_j-Cos[0]\mu_j)}{N_j^1/2}\overline{U_j}(\chi)$$
$$-\frac{2}{\delta(\chi)}\frac{d\delta(\chi)}{d\chi}\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}CP_{jk}U_j(\chi)+DP_k+EP_k\right)\overline{U_k}(\chi)-\frac{2}{\delta(\chi)}\frac{d\delta(\chi)}{d\chi}\frac{1}{3}\quad(32)$$

With the coefficients written below:

$$AP_{jk}=\int_0^1\tilde{\psi}_j(\eta)\,\tilde{\psi}_k(\eta)\,d\eta\ ;\ BP_k=\int_0^1\eta\tilde{\psi}_k(\eta)\,d\eta\ ;\ CP_{jk}=\int_0^1\tilde{\psi}_j(\eta)\,\eta\frac{d\tilde{\psi}_k(\eta)}{d\eta}d\eta\quad(33)$$

$$DP_k=\int_0^1\eta\eta\frac{d\tilde{\psi}_k(\eta)}{d\eta}d\eta\ ;\ EP_k=\int_0^1\eta\tilde{\psi}_k(\eta)\,d\eta\ ;\quad(34)$$

Energy Equation:

$$\begin{aligned}
&\delta_T(\chi)^2\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}(AT_{ijk}\overline{U_j}(\chi)+\frac{\delta_T(\chi)}{\delta(\chi)}ATF_{ik})\right)\frac{\overline{d\theta_k}(\chi)}{d\chi}+\frac{\delta_T(\chi)^2}{2}\frac{d\delta_T(\chi)}{d\delta(\chi)}\left(\sum_{j=1}^{\infty}(BT_{ij}\overline{U_j}(\chi)+\frac{\delta_T(\chi)}{\delta(\chi)}BTF_i\right)\\
&-\delta_T(\chi)\frac{d\delta_T(\chi)}{d\chi}\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}CT_{ijk}\overline{U_j}(\chi)+\frac{\delta_T(\chi)}{\delta(\chi)}CTF_{ik})\right)\overline{\theta_k}(\chi)+\delta_T(\chi)^2\delta(\chi)\sum_{j=1}^{\infty}HT_{ij}\frac{\overline{dU_j}(\chi)}{d\chi}\\
&+\delta_T(\chi)^2\frac{d\delta_T(\chi)}{d\chi}\left(\sum_{j=1}^{\infty}(DT_{ij}\overline{U_j}(\chi)+\frac{\delta_T(\chi)}{\delta(\chi)}DTF_i)\right)+\delta_T(\chi)\delta(\chi)\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}(ET_{ijk}\frac{\overline{dU_j}(\chi)}{d\chi}\right)\overline{\theta_k}(\chi)\\
&-\delta_T(\chi)\frac{d\delta(\chi)}{d\chi}\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}(FT_{ijk}\overline{U_j}(\chi)\right)\overline{\theta_k}(\chi)+\sum_{k=1}^{\infty}\left(GT1_{ik}-\frac{d\delta_T(\chi)}{d\chi}GT2_{ik}\right)\theta_k(\chi)\\
&\qquad-\delta_T(\chi)^2\frac{d\delta(\chi)}{d\chi}\left(\sum_{j=1}^{\infty}IT_{ij}\overline{U_j}(\chi)+(JT1_i-\frac{d\delta_T(\chi)}{d\chi}JT2_i)\right)=\frac{\Omega_i^2}{Pe}\overline{\theta_i}(\chi)+\frac{\delta_T(\chi)}{Pe}LT_i
\end{aligned}\tag{35}$$

With the coefficients written below:

$$AT_{ijk}=\int_0^1\tilde{\phi}_i(\eta_t)\,\tilde{\psi}_j(\eta)\,\tilde{\phi}_k(\eta_t)\,d\eta_t\ ;\ ATF_{ik}=\int_0^1\tilde{\phi}_i(\eta_t)\,\eta_t\tilde{\phi}_k(\eta)\,d\eta_t\ ;\ BT_{ij}=\int_0^1\tilde{\phi}_i(\eta_t)\,\tilde{\psi}_j(\eta)\,(1-\eta_t)^2d\eta_t \tag{36}$$

$$BTF_i=\int_0^1\tilde{\phi}_i(\eta_t)\,\eta(1-\eta_t)^2d\eta_t\ ;\ CT_{ijk}=\int_0^1\tilde{\phi}_i(\eta_t)\,\tilde{\psi}_j(\eta)\,\eta_t\frac{d\tilde{\phi}_k(\eta)}{d\eta_t}d\eta_t\ ;\ CTF_{ik}=\int_0^1\tilde{\phi}_i(\eta_t)\,\eta\eta_t\frac{d\tilde{\phi}_j(\eta_t)}{d\eta_t}d\eta_t \tag{37}$$

$$DTF_i=\int_0^1\tilde{\phi}_i(\eta_t)\,\eta\eta_t(1-\eta_t)d\eta_t\ ;\ DT_{ij}=\int_0^1\tilde{\phi}_i(\eta_t)\,\tilde{\psi}_j(\eta)\,\eta_t(1-\eta_t)d\eta_t\ ;\ ET_{ijk}=\int_0^1\tilde{\phi}_i(\eta_t)\,H_j(\eta)\frac{d\tilde{\phi}_k(\eta_t)}{d\eta_t}d\eta_t \tag{38}$$

$$FT_{ijk}=\int_0^1\tilde{\phi}_i(\eta_t)\,F_j(\eta)\frac{d\tilde{\phi}_k(\eta_t)}{d\eta_t}d\eta_t\ ;\ GT_{ij}=\int_0^1\tilde{\phi}_i(\eta_t)\,G(\eta)\frac{d\tilde{\phi}_k(\eta_t)}{d\eta_t}d\eta_t\ ;\ HT_i=\int_0^1\tilde{\phi}_i(\eta_t)\,H_j(\eta)(1-\eta_t)d\eta_t \tag{39}$$

$$IT_{ij}=\int_0^1\tilde{\phi}_i(\eta_t)\,F_j(\eta)(1-\eta_t)d\eta_t\ ;\ JT_{ij}=\int_0^1\tilde{\phi}_i(\eta_t)\,G(\eta)(1-\eta_t)d\eta_t\ ;\ LT_i=\int_0^1\tilde{\phi}_i(\eta_t)\,d\eta_t \tag{40}$$

The initial conditions of the hydrodynamic and thermal field are transformed and written as:

$$\overline{U_i}(0)=\int_0^1\tilde{\psi}_j(\eta)\,(U^*(0,\eta_t)-UF(\eta))\,d\eta=FF_i-HH_i\ ;\ \overline{\theta_j}(0)=\int_0^1\tilde{\phi}_j(\eta_t)\,(\theta^*(0,\eta_t)-TF(0,\eta_t))\,d\eta_t \tag{41}$$

In order to solve the coupled differential equation system, it is necessary to define the equations for the hydrodynamic and thermal layer thickness extracted from Naveira (2006) and Ozisik (1985), written as:

$$\frac{d\delta(\chi)}{d\chi}=2.50\ \chi^{-1/2}\ Re_L^{-1/2}\ ;\ \frac{d\delta_t(\chi)}{d\chi}=2.42048\ Pr^{-1/3}\ \chi^{1/2}\ Re_L^{-1/2} \tag{42}$$

Eqs. (27), (32), (35) and (42) form an infinite system of one-dimensional partial differential equations for the transformed potentials. For computational purposes this system is truncated to a sufficient large finite order, N (eigenvalues number), for the required convergence control. Once the transformed potentials are numerically computed, the inversion formula, Eqs. (25) - (26), is employed to reconstruct the filtered potentials, in explicit form in the transverse coordinate, and after adding the filtering solution, the dimensionless velocity and temperature distribution, is recovered everywhere within the region along the process.

4. RESULTS AND DISCUSSION

The Generalized Integral Transform Technique (GITT) proved to be an important tool to solve the proposed problem, being able to study and determine the thicknesses of the Thermal and Hydrodynamic boundary layer.

The developed Mathematica code incorporates all the symbolic and numerical computational steps in the solution procedure, being validated in several ways, including the comparison with the Blasius solution in the hydrodynamic field and with results obtained by Naveira (2006) for the thermal field.

In most of the works published in the literature, the velocity profile is determined using the Blasius solution and only the temperature profile is calculated. In this work, the hydrodynamic and thermal problems are solved simultaneously, using the momentum and continuity equation to determine the axial and radial velocity and the pressure gradient. Next, the energy equation is used to calculate the temperature field.

Figure 2 shows the behavior of the boundary layer thickness as a function of the position x(m), where we observe the obtained solution by present work is relatively close to the Blasius solution. The boundary layer thickness graph show even more deviation between proposed solution and the fifth-order and third-order Karman-Pohlhausen approximations. These two polynomial approximations were chosen because Eyo *et al.* (2012) performs the error calculations for the similarity parameter, reaching 7.18 and 3.58 %, respectively. The others polynomial approximations of different order showed larger errors.

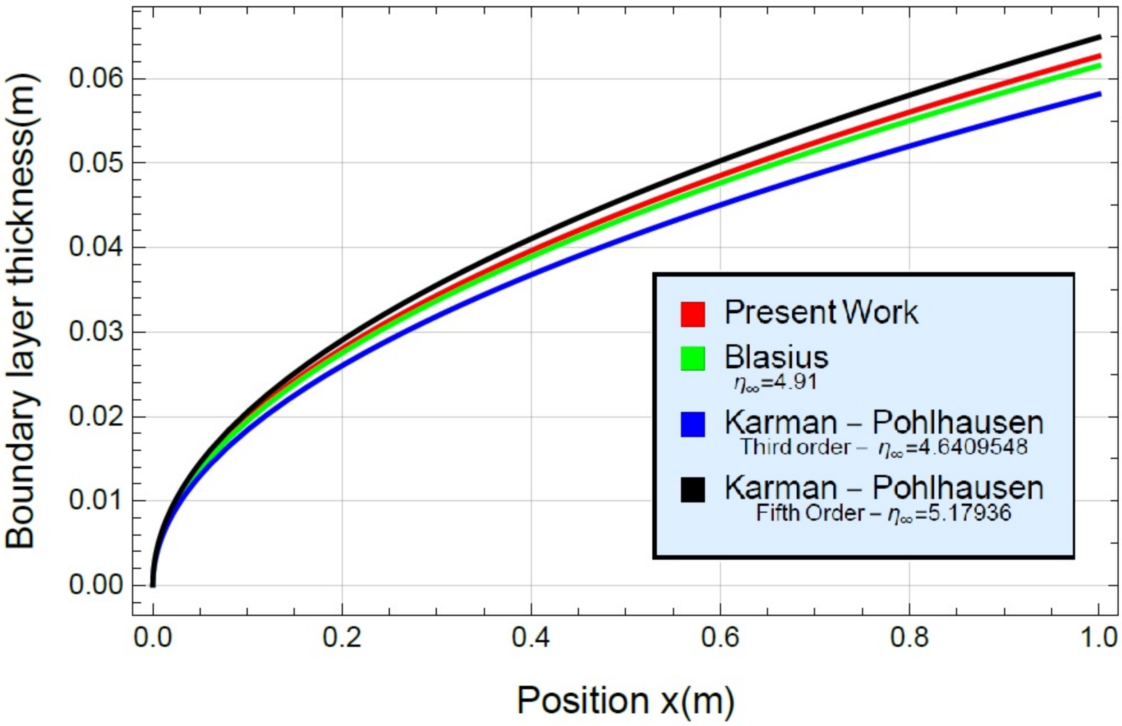

Figure 2. Behavior of the hydrodynamic boundary layer thickness as a function of the position x (m)

In order to compare the results of the temperature field we will use the data adopted by Naveira (2006) . Air was always the cooling fluid and the adopted numerical values in the simulations for the related governing parameters were:

$L = 0.1m;\ \ U_\infty = 1m/s;\ \ T_\infty = 20^oC;\ \ \alpha = 2.22x10^{-5}m^2/s;$

$k = 0.0262W/(m^oC);\ \ \nu = 1.57x10^{-5}m^2/s;\ \ \phi_{ref} = 100W/m^2;$

Figure 3 shows the behavior of the temperature as a function of the x and y position along the plate, using the Generalized Integral Transform Technique, presenting a good harmony with the solution obtained by Naveira (2006) . It is worth noting that as can be seen, in relation to the x position, the temperature distribution is similar to the profile of the thermal boundary layer thickness, since the y position is very small, there is no variation significant in this direction. This can be proved with the values presented in Table 2, where we fix a position in y and vary the position x.

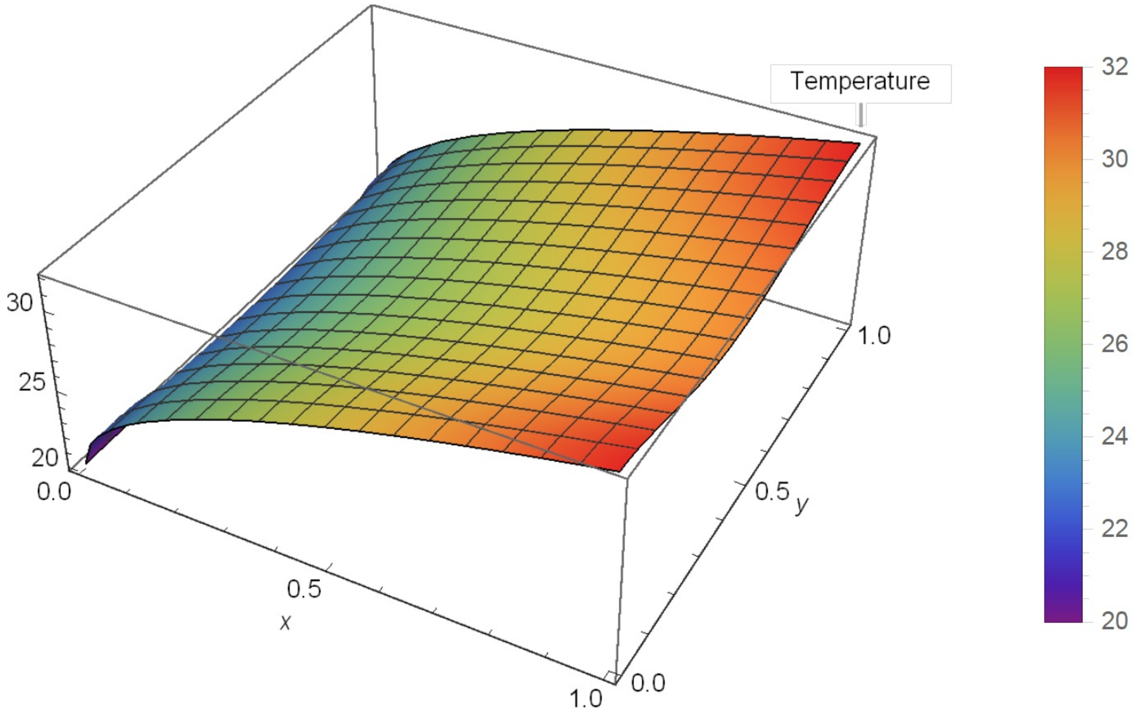

Figure 3. Temperature profile as a function of the x and y positions.

Figure 4 show the behavior of the thicknesses of the hydrodynamic and thermal boundary layers calculated by present work and compared with Naveira (2006), where it is verified that the thermal boundary layer thickness is greater hydrodynamic, since the number of Prandt adopted is less than 1.

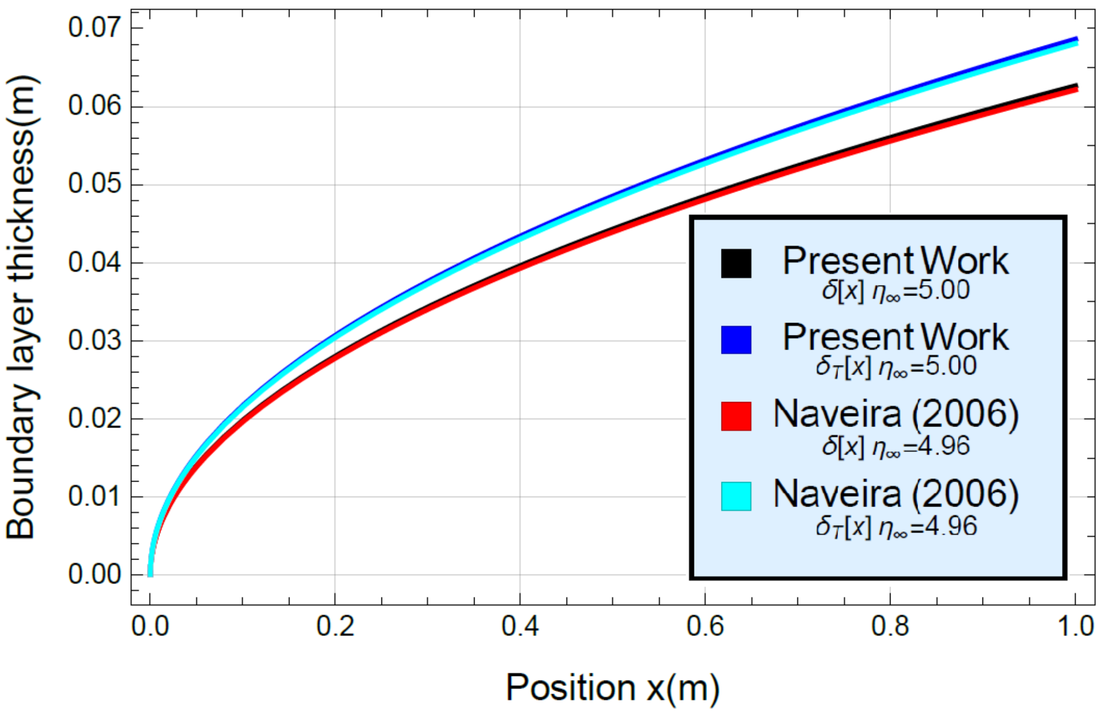

Figure 4. Behavior of the hydrodynamic and Thermal boundary layer thicknesses as a function of the position x (m)

Table 1 show the convergence of the thickness of the hydrodynamic boundary layer in comparison with Naveira (2006) and exact solutions, obtaining excellent results, with acceptable errors in the order of 1 %, which allows to prove that the model adopted using the NDSolve routine can be perfectly used. Comparing with Eyo *et al.* (2012) $\eta_\infty = 5.1793 \ \ and \ \ \eta_\infty = 4.6409$ we found errors in the order of 3.58 and 7.18 %, respectively. These errors obtained by Eyo *et al.* (2012) must to the applied methodology, where uses a integral equation for two dimensional laminar flow (Pohlhausen, 1921), finding η_∞ distant from the exact solution.

Table 1. Hydrodynamic boundary layer thickness as a function of position X(m) using NDSolve routine and comparing with others solutions that have different similarity parameters (η_∞).

x(m)	**Present Work** $\eta_\infty = 5.00$	**Exact Solution** $\eta_\infty = 5.00$	**Naveira (2006)** $\eta_\infty = 4.96$	**Eyo Et Al (2012)** $\eta_\infty = 5.1793$	**Eyo Et Al (2012)** $\eta_\infty = 4.6409$
1	0.0626492	0.0626498	0.0621486	0.0648964	0.0581503
0.9	0.0594343	0.0594348	0.0589594	0.0615662	0.0551662
0.8	0.0560352	0.0560357	0.0555874	0.0580451	0.0520112
0.7	0.0524161	0.0524166	0.0519973	0.0542963	0.0486520
0.6	0.0485278	0.0485283	0.0481401	0.0502686	0.0450430
0.5	0.0442996	0.0443001	0.0439457	0.0458887	0.0411185
0.4	0.0396227	0.0396232	0.0393062	0.0410441	0.0367775
0.3	0.0343142	0.0343147	0.0340402	0.0355452	0.0318502
0.2	0.0280173	0.0280179	0.0277937	0.0290226	0.0260056
0.1	0.0198111	0.0198116	0.0196531	0.0205221	0.0183887
$1.1x10^{-6}$	0.0000626	0.0000626	0.0000652	0.0000681	0.0000610

Table 2 show the temperature convergence, varying the number of eigenvalues up to 120 eigenvalues, obtaining convergence in the fourth decimal place. To validate the model, we compared it with results obtained by Naveira (2006) and by the Blasius solution showing good agreement.

Table 2. Convergence behavior of temperature distribution for different truncation orders, at y = 1 , x = 0.01 , 0.25 , 0.50 , 0.75 and 0.1 mm ; and comparing the results obtained with Naveira (2006) and Blasius solution.

Autovalores	**X=0.01mm**	**X=0.25mm**	**X=0.50mm**	**X=0.075mm**	**X=0.1mm**
10	23.6883	25.8318	28.2473	30.1009	31.6635
20	23.7036	25.8558	28.2814	30.1426	31.7117
40	23.7111	25.8678	28.2984	30.1634	31.7357
60	23.7137	25.8718	28.3040	30.1703	31.7437
80	23.7149	25.8738	28.3068	30.1738	31.7477
90	23.7154	25.8745	28.3078	30.1750	31.7490
100	23.7156	25.8750	28.3085	30.1758	31.7500
110	23.7159	25.8754	28.3091	30.1766	31.7509
120	23.7162	25.8758	28.3096	30.1772	31.7516
Naveira (2006)	23.7178	25.8717	28.3008	30.1652	31.7370
Blasius	23.7343	25.9044	28.3501	30.2267	31.8088

5. CONCLUSIONS

The present work was concerned with the solutions and physical interpretation of a steady-state conjugated conduction external convection and hydrodynamic problems, for laminar flow over a flat plate of non-negligible thickness. The solution for the determination of thermal and hydrodynamic boundary thickness allows the obtaining of results, that can be used the engineering level. The most important contribution of the research was to be able to determine the thickness of the hydrodynamic boundary layer without having to consider the Blasius hypothesis (similarity method) and the hydrodynamic and thermal problems was solved simultaneously. The results obtained in the determination of the hydrodynamic boundary layer thickness compared with the results of the exact solution were very close with an error of less than 1% , this corroborates for the validation of the proposed model, in addition we compared with the polynomial Karman-Pohlhausen approximations with third and fifth order (Eyo *et al.*, 2012) showing that these approximations cause a greater deviation from the exact analytical solution. The results obtained in the thermal field were also very satisfactory when compared with the results found in Naveira (2006).

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