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SHOOTING AND MATRIX METHOD APPLIED TO THE ANALYSIS OF STABILITY OF VISCOELASTIC JET FLOWS

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Abstract. *In the present work, linear stability theory is used to investigate the hydrodynamic stability of viscoelastic jet flows, particularly two-dimensional, planar, incompressible, submerged jets discharging from a nozzle into a medium of the same fluid as the jet. The constitutive stress relations will be those of the Oldroyd-B model. The instabilities found in jet flows are of the Kelvin-Helmholtz type, through two unstable modes: sinuous and varicose. This work aims to provide additional information on viscoelastic jet stability compared with the Newtonian jet results. This will be accomplished by comparing the two instability modes spatial growth rates and unstable frequency ranges. In addition, a comparison will be made between two ways of obtaining the stability analysis results, the Shooting method, through the numerical solution of the Orr-Sommerfeld equation, and solving an eigenvalue system through the matrix stability analysis. It was observed that the different approaches present similar results, that the matrix method corrects some spurious results obtained by the Shooting method, and that the non-Newtonian effect strongly influences the results for low Reynolds numbers.*

Keywords: *Jet flow, Oldroyd-B fluid, Linear Stability Theory, Orr-Sommerfeld equation, Matrix method.*

1. INTRODUCTION

Hydrodynamic stability analyzes the behaviour of a laminar flow subjected to a small amplitude perturbation. Once disturbed, the flow is stable if it returns to its laminar regime. Otherwise, if the disturbance grows, causing the flow to change to a different state, it is said to be unstable. The theory of stability is related to the mathematical analysis of the evolution of perturbations superimposed on a laminar base flow (González, 2020). The Orr-Sommerfeld equation governs the linear stability of hydrodynamic problems, and, for our case, this equation constitutes an eigenvalue problem (Danabasoglu and Biringen, 1990).

In many cases, correctly finding the eigenvalues of a system of differential equations can be a non-trivial undertaking. One of the problems encountered when using an iterative Shooting method is to determine the initial condition within an acceptable margin of error so that the proper boundary conditions for the particular problem can be satisfied. Another challenge is that small errors in the coefficient matrix associated with a system of differential equations can lead to relatively large errors in calculating the eigenvalues (Walker *et al.*, 2012). The collocation matrix method is easier to formulate, and the boundary conditions do not pose a problem; however, the numerical solution requires an efficient discretization. Since shear flows have steep gradients, it is natural to seek a discretization with clustering or stretching (Danabasoglu and Biringen, 1990).

The hydrodynamic stability theory has successfully investigated the stability characteristics of several non-Newtonian laminar flows, for example, the Poiseuille flow (Porteous and Denn, 1972; Brandi *et al.*, 2019; Zhang *et al.*, 2013). Jet flows are significant in fluid dynamics and engineering applications (Wang and Tan, 2010). Researchers have already been studying this type of flow: Rallison and Hinch (1995) considered the submerged elastic jet characterized by a parabolic profile flow for high Reynolds numbers. They found that the elasticity fully stabilizes the sinuous mode while the varicose mode is partially stabilized. In their linear stability analysis, the viscoelastic flow was modelled by the Oldroyd-B constitutive equation. The flow was more stable with the increase in the elastic effect but not fully stable. Zhang (2012) investigated the viscoelastic jet problem, in which the Oldroyd-B model was used for the constitutive equation. In this study, the result of the modal stability analysis in polymeric jets indicated that the elastic effect not only affected the hydrodynamic instability, increasing its critical Reynolds number, but also led to the emergence of a new instability mechanism, called elastic instability, for small Reynolds numbers. However, Zhang (2012) did not consider the distinction between varicose and sinuous modes. Sterza *et al.* (2020) investigated the hydrodynamic stability of viscoelastic jet flows, particularly incompressible, two-dimensional, planar, submerged jets discharging from a nozzle into a medium of the same fluid as the jet. The constitutive stress relations are based on the Oldroyd-B model. Their results showed that non-Newtonian effects are relevant at low values of Reynolds number. For varicose mode with high Weissenberg numbers, some results are not consistent and may be spurious results.

In the present work, linear stability theory is used to investigate the hydrodynamic stability of viscoelastic jet flows, particularly two-dimensional, planar, incompressible, submerged jets discharging from a nozzle into a medium of the same fluid as the jet. The constitutive stress relations are based on the Oldroyd-B model. The objective is to find and compare spatial amplification rates, using linear stability theory, through two approaches: the Shooting method and the matrix method. Moreover, verify if the matrix method is a good alternative to analyze the stability of this type of flow and, if possible, correct the spurious modes found in Sterza *et al.* (2020).

2. MATHEMATICAL FORMULATION

This paper assumes that the flow is two-dimensional, incompressible, isothermal, and with a non-Newtonian fluid. The conservation of mass and conservation of momentum equations governing the flow, in the dimensionless form, are given by

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\nabla p + \frac{\beta}{Re}\nabla^2 \mathbf{u} + \nabla \cdot \mathbf{T}, \tag{2}$$

where $\mathbf{u}$ denotes the velocity field, t is the time, p is the pressure, β is the dimensionless coefficient of the solvent viscosity, Re is the Reynolds number and $\mathbf{T}$ is the extra-stress tensor. Viscoelastic flows are studied in which the constitutive equation used is the Oldroyd-B model given in non-dimensional form by

$$\mathbf{T} + Wi\left[\frac{\partial \mathbf{T}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{T}) - \mathbf{T}(\nabla \mathbf{u})^T - (\nabla \mathbf{u})\mathbf{T}\right] = 2\frac{(1-\beta)}{Re}\mathbf{D}, \tag{3}$$

where Wi is the Weissenberg number and $\mathbf{D} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$ is the rate of deformation tensor.

We study viscoelastic plane jet flow where x and y represent the streamwise and wall-normal directions. The laminar base flow is assumed parallel. The streamwise jet base flow velocity component is the same used by Weder (2012) and the non-Newtonian extra-stress tensor components of the base flow are given by

$$U(y) = \frac{1}{2}\left[1 + \tanh\frac{R}{4\theta}\left(\frac{R}{y} - \frac{y}{R}\right)\right], \quad T_b^{yy} = 0, \quad T_b^{xy} = \frac{(1-\beta)}{Re}\frac{dU}{dy} \quad \text{and} \quad T_b^{xx} = 2WiT_b^{xy}\frac{dU}{dy}, \tag{4}$$

where R denotes the jet half-width and θ the momentum boundary layer thickness.

2.1 Linear Stability Theory

The linear stability theory analyzes the behaviour of a flow to perturbations of infinitesimal amplitude. The instantaneous flow is decomposed into a base flow and a disturbance, where the base flow is considered stationary and parallel. The disturbances are written as normal modes

$$\phi(x, y, t) = \overline{\phi}(y)e^{i(\alpha x - \omega t)}, \tag{5}$$

where $\overline{\phi}$ represents the magnitude and phase of the disturbances, $i = \sqrt{-1}$, $\alpha = \alpha_r + i\alpha_i$ is the wavenumber in the x direction (α_r) and the spatial growth rate (α_i) and ω is the angular frequency. Substituting the normal mode solution (5) into the disturbance Navier-Stokes and non-Newtonian extra-stress tensor equation, omitting the superscript from the notation we have

$$i\alpha u(y) + \frac{dv(y)}{dy} = 0, \tag{6}$$

$$-i\omega u + i\alpha uU + v\frac{\partial U}{\partial y} = -i\alpha p + \frac{\beta_n}{Re}\left[(i\alpha)^2 u + \frac{d^2u}{dy^2}\right] + i\alpha T^{xx} + \frac{dT^{xy}}{dy}, \tag{7}$$

$$-i\omega v + i\alpha vU = -\frac{dp}{dy} + \frac{\beta_n}{Re}\left[(i\alpha)^2 v + \frac{d^2v}{dy^2}\right] + i\alpha T^{xy} + \frac{dT^{yy}}{dy}, \tag{8}$$

$$T^{yy}\left(1 - i(\omega - \alpha U)Wi\right) = -Wi\left(\frac{dT_b^{yy}}{dy}v - 2i\alpha vT_b^{xy} - 2\frac{dv}{dy}T_b^{yy}\right) + \frac{2(1-\beta)}{Re}\frac{dv}{dy}, \tag{9}$$

$$T^{xy}\left(1 - i(\omega - \alpha U)Wi\right) = -Wi\left(v\frac{dT_b^{xy}}{dy} - i\alpha vT_b^{xx} - \frac{dU}{dy}T^{yy} - \frac{i}{\alpha}\frac{d^2v}{dy^2}T_b^{yy}\right) + \frac{(1-\beta)}{Re}\left(i\alpha v + \frac{i}{\alpha}\frac{d^2v}{dy^2}\right), \tag{10}$$

$$T^{xx}\left(1 - i(\omega - \alpha U)Wi\right) = -Wi\left(v\frac{dT_b^{xx}}{dy} + 2T_b^{xx}\frac{dv}{dy} - 2T_b^{xy}\frac{i}{\alpha}\frac{d^2v}{dy^2} - 2T^{xy}\frac{dU}{dy}\right) - \frac{2(1-\beta)}{Re}\frac{dv}{dy}. \tag{11}$$

3. NUMERICAL FORMULATION

For a two-dimensional plane jet, without body forces and with constant properties (ρ, η), the boundary conditions for the Orr-Sommerfeld equation are specified as follows (Kundu and Cohen, 2010)

$$\begin{aligned} \tilde{u} = \tilde{v} &= 0, \quad \text{for} \quad y \to \pm\infty, \\ \tilde{v} &= 0 \quad \text{for} \quad y = 0, \quad \text{(varicose mode)}, \\ \tilde{p} &= 0 \quad \text{for} \quad y = 0, \quad \text{(sinuous mode)}. \end{aligned}$$

Given the above conditions, this section presents the two approaches carried out for the stability analysis: the Shooting method and the matrix method.

3.1 Shooting Method

Rewriting the Eqs. (6) – (8) in a simplified way gives the modified Orr-Sommerfeld equation for Oldroyd-B fluid

$$\begin{aligned} &\frac{d^4v}{dy^4} - \left[i(\alpha U - \omega)\frac{Re}{\beta} + 2\alpha^2\right]\frac{d^2v}{dy^2} + \left[i\frac{Re\alpha}{\beta}\frac{d^2U}{dy^2} + i\frac{Re\alpha^2}{\beta}(\alpha U - \omega) + \alpha^4\right]v + \\ &+\frac{Re}{\beta}\left[\alpha^2\frac{dT^{xx}}{dy} - i\alpha\frac{d^2T^{xy}}{dy^2} - i\alpha^3T^{xy} - \alpha^2\frac{dT^{yy}}{dy}\right] = 0. \end{aligned} \tag{12}$$

This equation is known as the Orr-Sommerfeld equation for a viscoelastic fluid Oldroyd-B type (Souza *et al.*, 2016). Its solution corresponds to an eigenvalue problem that was solved numerically by a Shooting method, (Drazin and Reid, 2004) integrating the Orr-Sommerfeld equations from $\pm\infty$ to the centre of the jet and using a Gram-Schmidt orthonormalization procedure to ensure the eigenvectors remain orthogonal. The solution is directly linked to the values of α, ω, Re, β, and Wi, and depends on the base flow velocity profile in question (Brandi *et al.*, 2017). Since $u(y \to \pm\infty) = 0$, all derivatives of u with respect to y must also be zero at $y \to \pm\infty$. Considering Eqs. (12) – (11) together with appropriate boundary conditions one can obtain results for linear stability analysis. In order to accelerate the convergence, mesh stretching was used because the most significant gradients in the base flow velocity profile is close to the jet.

3.2 Matrix Stability Analysis

It is common for the system of Eqs. (6) – (11) to be used to derive the Orr-Sommerfeld equation, as seem in the previous subsection. However, for the second approach used to analyse the stability, matrices are used. Matrix stability analysis allows determining the stability properties of any algorithm combined with its boundary conditions. In other words, it is necessary to write the Eqs. (6) – (11) in matrix form, obtaining the matrices L and F, where

$$L[u \quad \alpha u \quad v \quad \alpha v \quad p \quad T^{xx} \quad T^{yy} \quad T^{xy}]^\intercal = \alpha F[u \quad \alpha u \quad v \quad \alpha v \quad p \quad T^{xx} \quad T^{yy} \quad T^{xy}]^\intercal,$$

for spatial analysis. Note that the L and F matrices are constructed with the elements of the Eqs. (6) – (11), that is, each row of the L and F matrices corresponds to one of the equations mentioned above, and it is enough to find the corresponding eigenvalues and eigenvectors. In this case, it was necessary to use the points and the Chebyshev differentiation matrices, as they were necessary for calculating the derivatives. Furthermore, it should be noted that the matrix stability analysis is solved for perturbations in a velocity profile $\tanh$, which presents high gradients at the jet and extend to infinite in the y direction. On the other hand, the Chebyshev points are not evenly spaced and belong to the range $[-1, 1]$. To solve this, a mapping function was introduced (Reddy *et al.*, 1999; Juniper *et al.*, 2014):

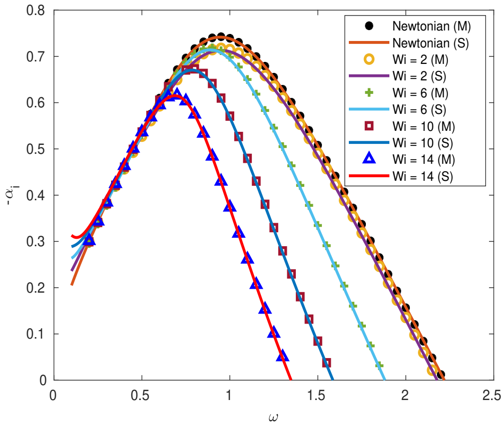
$$y = -\frac{l\tilde{y}}{\sqrt{1 + s - \tilde{y}^2}}, \tag{13}$$

where $l = 0.8$, $s = (l/s_\infty)^2$, $s_\infty = 8$ and $\tilde{y} \in [-1, 1]$ refers to the Chebyshev points. When using mapping, the domain becomes $y \in [-s_\infty, s_\infty]$. The constant l is the refinement parameter chosen from the tests performed. The numerical code was implemented in the Matlab software, and the eigenvalues and eigenvectors were obtained using the Matlab command `eig`.

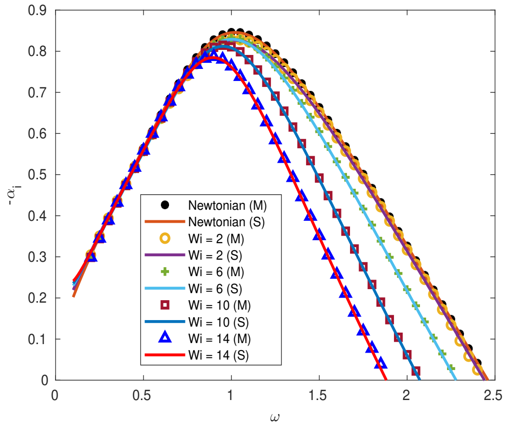
4. RESULTS

According to Sterza *et al.* (2020), non-Newtonian effects are more relevant for low Reynolds numbers, then the tests were performed with the following parameters: $Re = 250$ and 500, $\beta = 0.3, 0.5$ and 0.7 and $Wi = 2, 6, 10$ and 14. Furthermore, the base flow used was the one presented in Eq. (4) being the velocity profile with $R = 1$ and $\theta = 0.1$.

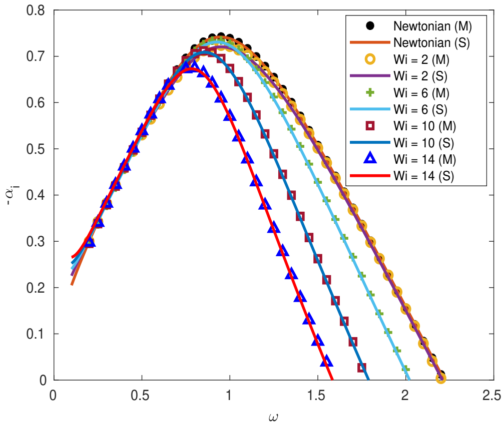
For the Shooting method we used 8201 points in the y direction, and $y \in [-4, 4]$. While in the matrix method, we used 551 numbers of Chebyshev collocation points and $y \in [-8, 8]$. We investigated the growth rate α_i in the spatial stability analysis of the jet flow, with $\omega \in \mathbb{R}^+$.



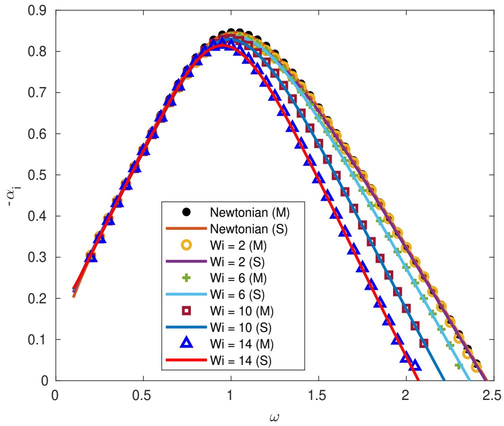
(a) $Re = 250$ and $\beta = 0.3$.



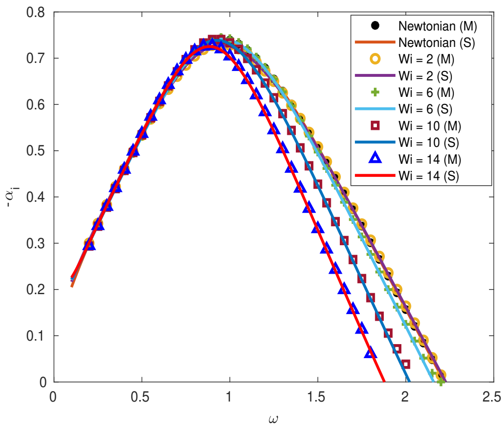
(b) $Re = 500$ and $\beta = 0.3$.



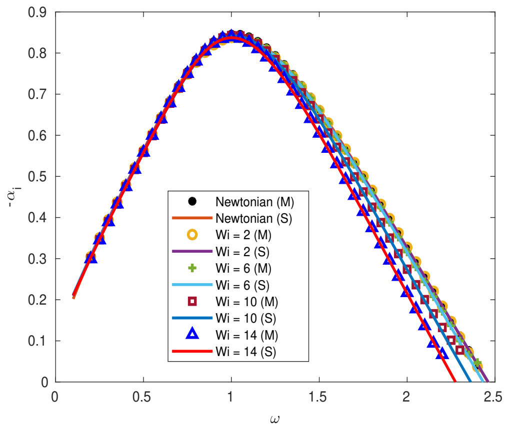
(c) $Re = 250$ and $\beta = 0.5$.



(d) $Re = 500$ and $\beta = 0.5$.

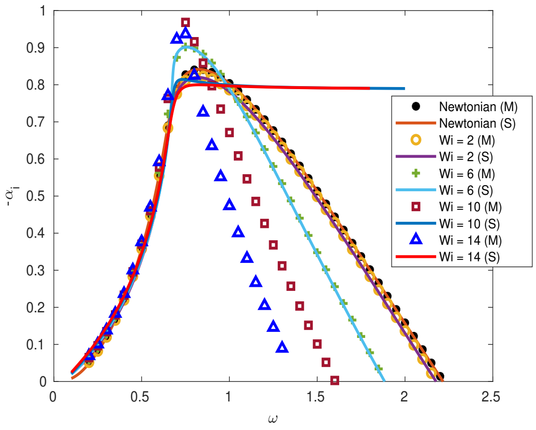


(e) $Re = 250$ and $\beta = 0.7$.

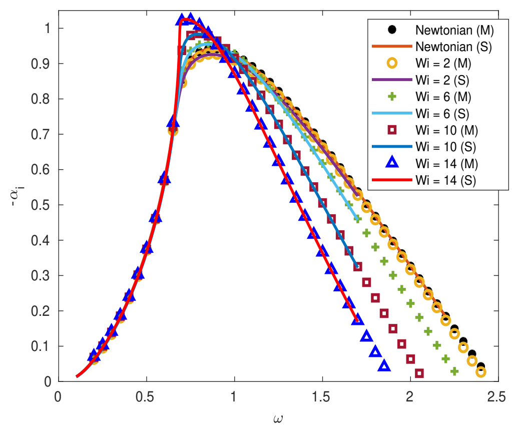


(f) $Re = 500$ and $\beta = 0.7$.

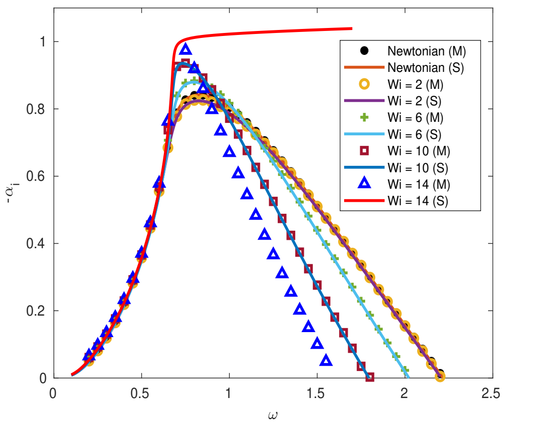
Figure 1. Sinuous mode growth rates for different Weissenberg numbers, $Re = 250$ and 500 and $\beta = 0.3, 0.5$ and 0.7.



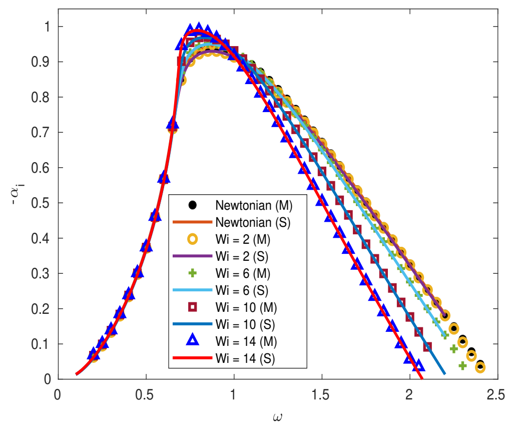
(a) $Re = 250$ and $\beta = 0.3$.



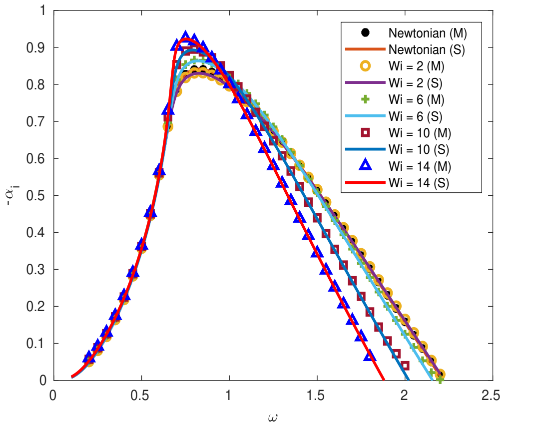
(b) $Re = 500$ and $\beta = 0.3$.



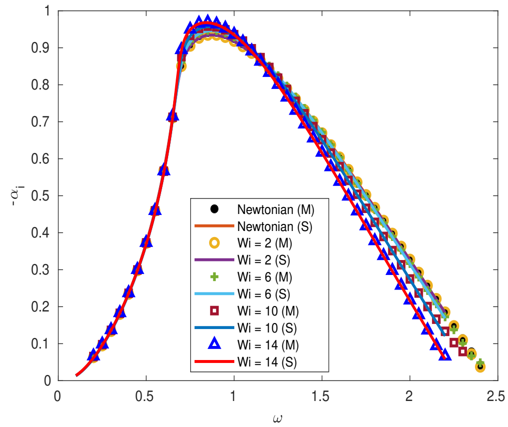
(c) $Re = 250$ and $\beta = 0.5$.



(d) $Re = 500$ and $\beta = 0.5$.



(e) $Re = 250$ and $\beta = 0.7$.



(f) $Re = 500$ and $\beta = 0.7$.

Figure 2. Varicose mode growth rates for different Weissenberg numbers, $Re = 250$ and 500 and $\beta = 0.3, 0.5$ and 0.7.

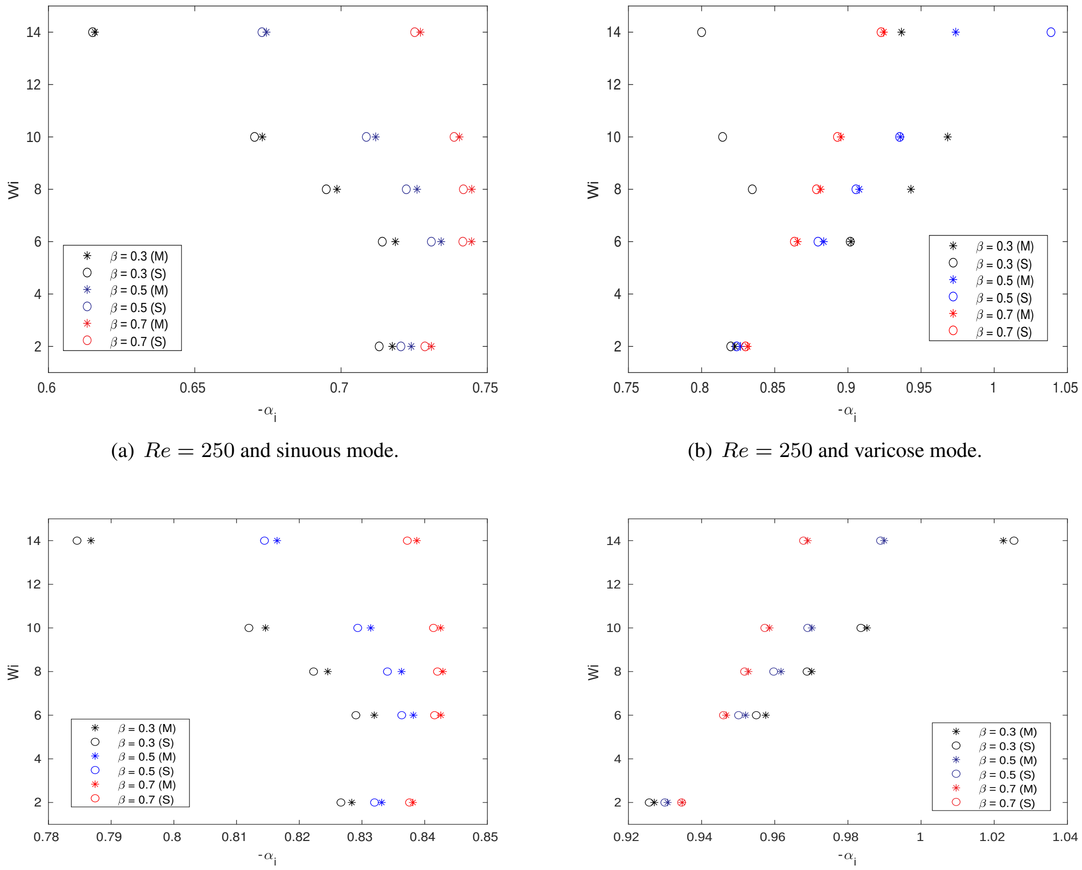
Figure 1 shows the growth rates for the sinuous mode for the dimensionless parameters: $Re = 250$ and 500, $\beta = 0.3, 0.5$ and 0.7 and different Weissenberg numbers, $Wi = 2, 6, 10$ and 14 and, also, for Newtonian fluid (considering $\beta = 1$ and $Wi = 0$). Figure 2 shows the results for varicose mode, using the same parameters. In the Figures, the Shooting method and matrix method results are represented by (S) and (M), respectively.

It is observed that the highest growth rate of the varicose mode is about 10% higher than that of the sinuous mode. However, its stability characteristics are very similar, with the same unstable frequency range (Sterza, 2020). Furthermore, the behaviour of the two methods studied is similar.

Note that the results obtained by the matrix and the Shooting methods are in agreement since they present very similar results. It is noteworthy that the spurious modes captured by the Shooting method for the highest Weissenberg values are corrected by the matrix method, as can be seen in Figs. 2(a) and 2(c).

In addition, it is possible to verify the non-Newtonian effect on the results. Starting with the elastic effects: as the Weissenberg number increases, the unstable frequency range decreases and, for the sinuous mode, the maximum amplification rate decreases, while for the varicose mode, it increases. Regarding viscosity: as the Reynolds number increases, the unstable frequency range also increases; the same can be observed when the constant β increases.

Figure 3 shows the variation in the maximum growth rate with Weissenberg numbers for different β values. For the sinuous mode, the growth rate initially increases with Wi, but above a specific value, it starts to decrease. Furthermore, for the varicose mode, the value only increases with increasing the Weissenberg number, except when capturing spurious modes.



(a) $Re = 250$ and sinuous mode.

(b) $Re = 250$ and varicose mode.

(c) $Re = 500$ and sinuous mode.

(d) $Re = 500$ and varicose mode.

Figure 3. Maximum growth rate variation with the Weissenberg number for different β for $Re = 250$ and 500.

5. CONCLUSIONS

The present work analyzed the stability of two-dimensional jet flows with viscoelastic fluids using the constitutive equation of the Oldroyd-B model. The analysis was performed with two different approaches, considering spatial disturbances for stability analysis: solving the Orr-Sommerfeld equation using the Shooting method; and solving the eigenvalue problem of the stability analysis matrix. The objective of this work was to compare the amplification rates obtained through the two approaches and verify if the matrix method would correct the spurious modes captured by the Shooting method.

The results obtained by the stability matrix are consistent with those obtained by the Shooting method, and the spurious modes, for $\beta = 0.3$ (Fig. 2(a)) and 0.5 (Fig. 2(c)) in varicose mode, were corrected. In addition, non-Newtonian effects can be observed through the dimensionless parameters tested; for example, when the amount of the polymer concentration in the fluid and Weissenberg number are decreased, and the Reynolds number is increased, the non-Newtonian effect on the stability is reduced.

6. ACKNOWLEDGEMENTS

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