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RE-EAVALUATION OF LINEAR STABILITY ANALYSIS OF SAINT-VENANT EQUATIONS: THE CONVECTIVE NATURE OF ROLL WAVES INSTABILITIES

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Abstract. *Fluid flows may, eventually, can be presented in sufficient condition for certain structures to transition to turbulence to appear, the so-called hydrodynamic instabilities. The relevance of these instabilities is manifested as they lead to more dissipative flow configurations or even bringing orthogonal variations in properties which one wishes to control. Therefore, it is necessary to understand how these instabilities evolve, in order to control and predict the behavior of such flows. In this sense, the objective of this work is to diagnose, with a view to the application in engineering, certain types of instabilities namely: roll waves occurring in turbulent flow over an inclined channel. From linear stability analyzes, at first, a mathematical model of the literature is recovered to define the formation criteria of roll waves. In a second step, it is shown mathematically that the instability criteria are based on the Froude number, this occurs when the specific mass variation is large, given a slender interface.*
Keywords: *Roll waves, Instabilities, Flows.*

1. INTRODUCTION

Fluid flows are essentially an interdisciplinary subject, which have wide areas of application, such as: flows through obstacles (bridge pillars, buildings, wind farms), geophysical flows (river floods and overflows, landslides), debris and mud flows, snow avalanches, volcanic eruptions), industrial flows (dam spillways, pipe bank), among others. The behavior of these flows is subject to several conditions, principles, and laws which make scene to inumerous problems. In these scenarios, flows can be presented in sufficient conditions for certain structures of transition to turbulence to appear, the so-called hydrodynamic instabilities. Therefore, it is necessary to obtain control of these instabilities in order to know, control, and predict the behavior of such flows.

Hydrodynamic instabilities appear in the neighborhood of the steady and uniform regime established by the flow in an attempt to balance the forces involved. Usually, such a balance of forces is constituted between active ones - field forces and surface forces, and resistive forces - usually associated with viscous friction and depending on the physical and rheological properties of the flowing fluid. The conditions for the flow to become unstable, and for the propagation, amplification and/or maintenance of small disturbances/instabilities to occur, will depend on the flow conditions, and on the intrinsic characteristics of the disturbances made.

In this scenario, roll waves instabilities can arise from the unbalance of gravitational and viscous forces acting on the flow, which can amplify small disturbances and culminate in a well-defined periodic variation on the free surface of the flow similar to a shock wave (Balmforth and Mandre, 2004; Noble, 2007; Ferreira, 2013). Due to its oscillatory nature, the presence of instabilities in natural and artificial flows could potentialize the unpredictability of flows and losses of many levels (civil, urban, and social infrastructure) arising from it.

Several researchers have made efforts in an attempt to find the necessary and sufficient mathematical conditions for such instabilities to occur. The roll waves has been studied in depth by the scientific community since Jeffreys (1925), and have been improved with Ishihara *et al.* (1954); Vedernikov (1945); Balmforth and Mandre (2004); Noble (2007); Di Cristo and Vacca (2005), for example. The dynamic characteristics of these instabilities as well as their conditions of propagation, however, still deserve efforts from the scientific community, in favor of their better knowledge.

Thus, this work comes with the motivation to reevaluate some of the classic cases of literature and to improve the analyzes already conducted, with the purpose of better understanding the conditions of maintenance and appearance of roll waves. The specific objective of this work is to conduct a linear stability analysis on the governing equations system (shallow waters, Saint-Venant equations), and to culminate with information on temporal branches and convectivity, following the work of Di Cristo and Vacca (2005). In this work, it is shown that the convectivity is respected, despite the discrepancy found in the equation with that of the literature.

2. METODOLOGY

To carry out this work, we used the one-dimensional Saint-Venant mathematical model that describes the behavior of free surface flows when the flowing fluid is Newtonian, when the regime is transient and turbulent. Such a system represents the conservation of the flow mass and momentum. In order to assess the appearance of roll wave instabilities, it was sought, through a mathematical tool, to develop linear analysis of stability.

The linear stability theory is a tool that uses the linearization of the governing equations of the problem and the application of an infinitesimal disturbance to the base flow, followed by the asymptotic expansion with respect to a small wavenumber. This tool has made it possible to obtain important information on problems of various kinds. Some of the relevant information stands out as: knowledge of the problem's stability domain (marginal curve of stability); the evolution of disturbances (amplificiation or evanescence); identification of characteristics of developed (steady) instabilities. Several authors bring this methodology, among them Ferreira *et al.* (2011); Pascal *et al.* (2013); Di Cristo *et al.* (2010); Noble (2007); Ferreira (2013).

The main concepts that involve such analysis can be exemplified through the Ginzburg-Landau equation (Eq. (1)), as well explained by Huerre and Rossi (1998b),

$$\frac{\partial \psi}{\partial t} + U\frac{\partial \psi}{\partial x} = \mu\psi + \frac{\partial^2 \psi}{\partial x^2} - |\psi|^2\psi \tag{1}$$

where $\psi(x,\ t)$ is the target variable that describes the phenomenon and depends on time t and space x, and U and μ are controlling parameters for the phenomenon.

In order to carry out the linear stability analysis, the equations are linearized disregarding terms of an order greater than or equal to 2, and can be rewritten as

$$\frac{\partial \psi}{\partial t} - \mathbf{L}(\nabla, \psi_0(x,t), R)\psi = 0 \tag{2}$$

where $\mathbf{L}$ is a linear operator, R is a set of phenomenological parameters (in this case U and μ), $\psi_0(x,\ t)$ is the state of equilibrium, solution of the problem. This means that for a specific set of parameters R, the solution will converge asymptotically to $\psi_0(x,\ t)$ if small disturbances are applied to the system, i.e. the system is Lyapunov stable.

For this work, we desire to know how the function $\psi(x,\ t)$ evolves when the solution assumes the so-called normal modes

$$\psi(x,t) = A\exp[i(kx - \omega t)] \tag{3}$$

where A is the amplitude, k is the wavenumber, and ω is the frequency of the solution. Thus, it is possible to rewrite the base equation of a given problem, applying Eq. (3) in Eq. (2) and obtaining the dispersion equation to the problem:

$$D(k,\omega,R) = 0 \tag{4}$$

From this equation, it is possible to observe the temporal evolution of small disturbances, making the consideration that k is real, or still, observe the spatial evolution, making the consideration that ω is real (Huerre and Monkewitz, 1990; Briggs, 1964). It is intended here to reevaluate such analyzes for a specific problem of the literature brought by Di Cristo *et al.* (2010), and to observe the growth rates of instabilities, celerity of such waves, as well as their convectivity. The entire mathematical procedure will be considered as a result of this work, and follows in the next section.

3. RESULTS

3.1 Mathematical model

Mathematical modeling consists of the art of transforming problems of reality into mathematical problems that arise in the most diverse areas, whether they come from physics, engineering, among others. In addition, the mathematics applied to these problems has helped us in a decisive way for the understanding of natural phenomena, allowing the representation of the concepts and processes involved, and also providing the understanding of aspects of problems that are not easily revealed.

However, to model fluid flows on a free surface, a set of equations has been studied for more than three decades, and constitute the basis for the hydrodynamic models of shear flows, among which we highlight:

- Mass conservation equation, also called the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \tag{5}$$

- Equation of momentum conservation:

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) = -\nabla p + \nabla\cdot\overline{\tau} + \rho\vec{g} \tag{6}$$

These are partial differential equations where the independent variables are the spatial coordinates and time t, and the following dependent variables: velocity field $\vec{u} = (u,\ v,\ w,\ t)$, pressure field $p(x,\ y,\ z,\ t)$, viscous tensor $\overline{\tau}(x,\ y,\ z,\ t)$. Among other parameters of interest: mass specifies ρ, gravity acceleration $\vec{g}$.

In addition, to determine the most appropriate mathematical model capable of representing the desired phenomenon for the flow, some hypotheses can be considered, for example: steady or transient flow, uniform or non-uniform, laminar, turbulent, incompressible, among others. These hypotheses transform the system and produce a set of partial differential equations that can only be solved using simplifying hypotheses.

In the case of problems that model shallow water the simplifying hypotheses are (Ferreira, 2013):

- the characteristic length (L) must be greater than the flow depth (h). This hypothesis shows that only long waves, that is, waves where the length is greater than the height, are taken into account;
- the width of the channel (d) is much greater than the height of the column (h) of the fluid, thus, it is possible to simplify the calculation of the resistance forces, ignoring the contribution of sidewalls, considering only the fluid tension with the bottom of the channel, rendering the problem two-dimensional.

A mathematical model of shallow waters, Saint-Venant equations, is presented by Di Cristo and Vacca (2005), in which the one-dimensional governing equations are given by:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} + \frac{\tau_b}{\rho h} = gS_0 \tag{7}$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0, \tag{8}$$

where x is the longitudinal coordinate of the flow, t the time, g is the acceleration of gravity, ρ the specific mass, S_0 the slope of the bed and τ_b the shear stress.

To mathematically evaluate the problem, the Eq. (7) and Eq. (8) system is rewritten in dimensionless variables. For this process, the scales of the various parameters must be chosen appropriately in order to provide coefficients that can represent proportions between flow characteristics. The dimensionless variables (denoted by $\tilde{}$) and scales were chosen and given as follows:

- Length scale: $\tilde{x} = \dfrac{xS_0}{h_0}$ and $\tilde{h}' = \dfrac{h'}{h_0}$
- Velocity scale: $\tilde{u}' = \dfrac{u'}{u_0}$
- Time scale: $\tilde{t} = \dfrac{tS_0u_0}{h_0}$
- Froude number: $F = \dfrac{u_0}{\sqrt{gh_0}}$

where u_0 and h_0 are the flow velocity and height for the steady and uniform solution, and u' and h' are disturbed quantities of flow velocity and height.

Substituting these variables in the equations Eq. (7) and Eq. (8) and manipulating them algebraically, the result is expressed by the following equations:

$$\frac{\partial \tilde{u}'}{\partial \tilde{t}} + \frac{\partial \tilde{u}'}{\partial \tilde{x}} + \frac{1}{F^2}\frac{\partial \tilde{h}'}{\partial \tilde{x}} + \frac{1}{F^2}\left(2\tilde{u}' - \tilde{h}'\right) = 0, \tag{9}$$

$$\frac{\partial \tilde{h}'}{\partial \tilde{t}} + \frac{\partial \tilde{h}'}{\partial \tilde{x}} + \frac{\partial \tilde{u}'}{\partial \tilde{x}} = 0 \tag{10}$$

Deriving Eq. (9) in relation to the variable $\tilde{x}$ and Eq. (10) in relation to the variable $\tilde{x}$ and $\tilde{t}$ respectively, get:

$$\frac{\partial^2 \tilde{u}'}{\partial \tilde{x}\partial \tilde{t}} + \frac{\partial^2 \tilde{u}'}{\partial \tilde{x}^2} + \frac{1}{F^2}\frac{\partial^2 \tilde{h}'}{\partial \tilde{x}^2} + \frac{1}{F^2}\left(2\frac{\partial \tilde{u}'}{\partial \tilde{x}} - \frac{\partial \tilde{h}'}{\partial \tilde{x}}\right) = 0, \tag{11}$$

$$\frac{\partial^2 \tilde{u}'}{\partial \tilde{x}^2} = -\frac{\partial^2 \tilde{h}'}{\partial \tilde{x}\partial \tilde{t}} - \frac{\partial^2 \tilde{h}'}{\partial \tilde{x}^2} \tag{12}$$

and

$$\frac{\partial^2 \tilde{u}'}{\partial \tilde{x}\partial \tilde{t}} = -\frac{\partial^2 \tilde{h}'}{\partial \tilde{t}^2} - \frac{\partial^2 \tilde{h}'}{\partial \tilde{x}\partial \tilde{t}} \tag{13}$$

Substituting the equations Eq. (12) and Eq. (13) in Eq. (11) get the following equation:

$$\frac{\partial^2 \widetilde{h}'}{\partial \tilde{t}^2} + 2\frac{\partial^2 \widetilde{h}'}{\partial \tilde{t}\partial \tilde{x}} + \left(1 - \frac{1}{F^2}\right)\frac{\partial^2 \widetilde{h}'}{\partial \tilde{x}^2} + \frac{1}{F^2}\left(3\frac{\partial \widetilde{h}'}{\partial \tilde{x}} + 2\frac{\partial \widetilde{h}'}{\partial \tilde{t}}\right) = 0 \tag{14}$$

Following the theory of classical linear stability, the perturbation $\tilde{h}'$ is decomposed into elementary waves like $\exp i(k\tilde{x} - \omega\tilde{t})$, with $k = k_r + k_i$ the dimensionless complex wavenumber and $\omega = \omega_r + i\omega_i$ the dimensionless complex frequency. The substitution of such a disturbance in the equation Eq. (14) results in the quadratic dispersion equation, Eq. (15),

$$D(k,\ \omega,\ F) = F^2 i(\omega - k)^2 - k^2 i - 2\omega + 3k = 0 \tag{15}$$

From the equation Eq. (15) the analytical form of the solution can be deduce, also called time branches $\omega(k)$. So, considering $k = k_r$

$$\omega_{\pm}(k) = k - \frac{i}{F^2} \pm \sqrt{\frac{k}{F^2}(k+i) - \frac{1}{F^4}}, \tag{16}$$

These temporal branches, Eq. (16), confirm the ones presented by Di Cristo and Vacca (2005), and can be empoloyed to study the evolution of the solution over time and space. Equation (16) can be employed to obtain the marginal curve of stability observing the growth rate of small disturbances as shows Eq. (17),

$$Im\left\{\omega_{\pm}(k)\right\} > 0 \tag{17}$$

The existence of instabilities is represented by the imaginary part of the branches $\omega_{\pm}(k)$, composing the rate of temporal growth of the instabilities through a spatial disturbance (Briggs, 1964; Huerre and Rossi, 1998a; Fiorot and Maciel, 2019). In these conditions, when Eq. (17) is verified, the domain favorable to the propagation of instabilities can be obtained. Applying Eq. (17) in Eq. (16), have to

$$F > 2 \tag{18}$$

Thus, through this analysis of temporal linear stability, it is identified that the control parameters F, that is, there is a minimum Froude number, $F_c = 2$, above which instabilities will be amplified in this system. In order to validate these observations, Fig. 1 illustrates the growth rate of disturbances ω_i as a function of the real wavenumbers k_r for various Froude numbers.

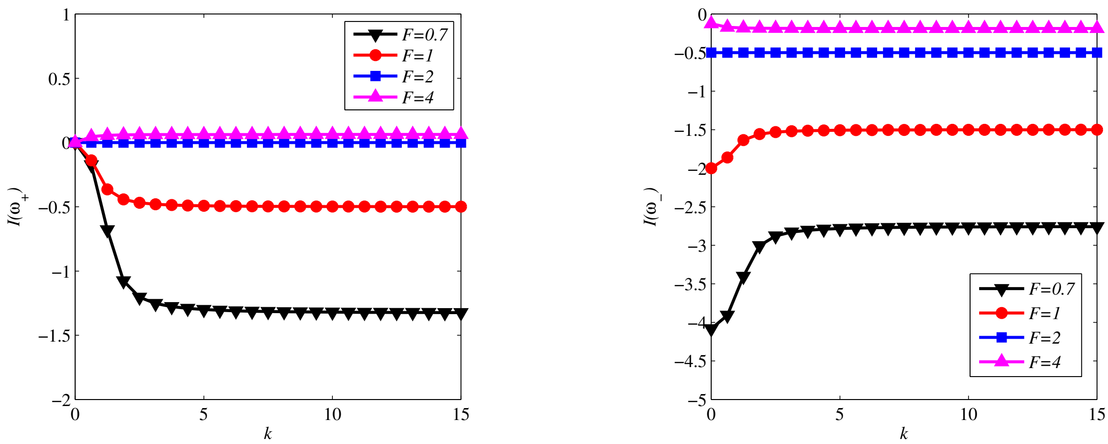

Figure 1. Growth rate of instabilities ω_i as a function of the wavenumber for different Froude numbers.

It can be seen in Fig. 1 that for different numbers of Froude two temporal branches coexist, this is due to the quadratic characteristic of the dispersion equation (Eq. (15)). The increase in the instability rate occurs for Froude numbers greater than 2, as both branches are positive, that is, instability is growing.

Indeed, it is worth noting that this development corroborates the classic developments already brought by the literature as Jeffreys (1925) which, using the Saint-Venant model, disregarding the diffusive effects, culminated in a critical value of Froude's number, $F_c = 2$, for turbulent flows of high Reynolds number, this being one of the first to systematize the roll wave formation criterion. It should also be noted that the base model used by Di Cristo and Vacca (2005) is also verified. This criterion is of paramount importance since it establishes a minimum value for the relationship between inertial effects and the effect of gravity for which this type of instability may arise (Fiorot, 2012).

Another important parameter that can be considered for this analysis is the wave propagation velocity (or celerity) U. It has been shown that the celerity can also be a parameter of control of the instabilities (Needham and Merkin, 1984). The propagation velocity is calculated through the real part of the branches $\omega_{\pm}(k)$ and is determined by

$$U = \frac{Re\,\{\omega_{\pm}(k)\}}{k}, \tag{19}$$

Fig. 2 illustrates the results of the calculations.

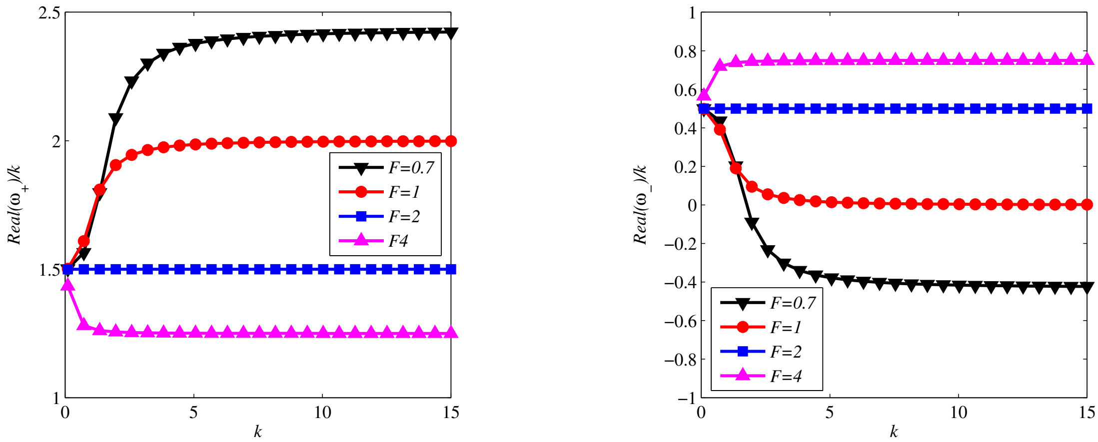

Figure 2. Velocity of propagation of disturbances as a function of the wave number for different Froude numbers.

This result shows that growing waves (those with $F > 2$) propagate more slowly than evanescent waves (thoese with $F \leq 2$), and for $F = 2$ the slope of the curve changes sense in relation to the other curves on which the tests were made. As we have performed a long wave approach, where the valid waves have low wavenumbers, the limit values for wave propagation velocities were estimated depending on the Froude number, as shows Fig. 3.

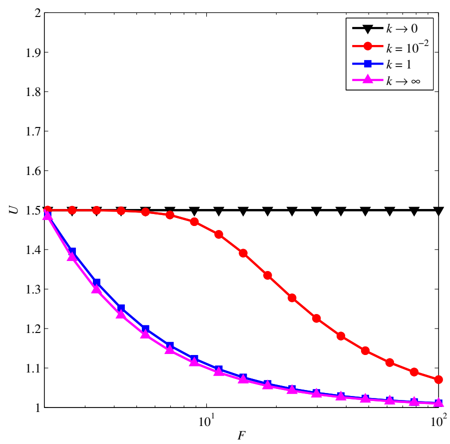

Figure 3. Wave propagation velocity as a function of the Froude number for different wavenumbers k.

Figure 3 shows the celerity U have a strong dependance on the Froude number only for high wavenumbers. For low wavenumber, U assumes a constant value. We higlight that the limits of x-axis were fixed for growing waves $(F > 2)$ and that for great Froude numbers, the physical meaning can be lost. The observations allow the following summarization:

- for $k \rightarrow 0$, $\lim\limits_{k \rightarrow 0} U = \frac{3}{2}$;

- for $k \rightarrow \infty$, $\lim_{k \rightarrow \infty} U = 1$.

Needham and Merkin (1984) reported the mathematical observation that the parameter U, when valid for roll waves generation $1 < U < \frac{3}{2}$, is intrinsically related to the flow properties and wave characteristics. It can lead to a change in the point of equilibrium of the problem in a way that a Hopf bifurcation can be obtained.

In the Saint-Venant model, Eq. (7) and (8), from which the dispersion equation (Eq. (15)) was obtained, there are only two spatial branches. Briggs (1964) proved that the absolute character of instability can be defined by looking for complex frequencies ω^0 for two (or more) space branches $k(\omega)$. Based on Eq. (15), following Briggs (1964); Huerre and Monkewitz (1990), we corroborate the equation obtained by Di Cristo and Vacca (2005):

$$k(\omega) = \frac{2F^2\omega + 3i \pm \sqrt{4F^2\omega(\omega + i) - 9 + 8i\omega}}{2\left(F^2 - 1\right)} \tag{20}$$

The positive definitiveness of ω_i^0 is a necessary condition for the absolute character of instability. The sufficient condition comes analyzing from ω values sufficiently large, that is, $\omega_i > \omega_i^{\max}$. From the $k(\omega)$ space branches that coalesce to $\omega = \omega^0$, the instability will be absolute if for $\omega_i > \omega_i^{\max}$ at least two of these branches are placed on opposite sides of the real k axis. As the merging points of the space branches correspond to the saddle points of the temporal branches $\omega(k)$, the necessary condition of the Briggs (1964) criterion can be analyzed alternatively by looking for the complex wave number k^0, and to determine this it is enough that $\frac{d\omega}{dk} = 0$.

Thus, the rate of change of $\omega(k)$ can be calculated from Eq. (16) as follows:

$$\frac{d\omega}{dk} = 1 \pm \frac{2k + i}{2\sqrt{F^2(k^2 + ik) - 1}} = 0, \tag{21}$$

Solving Eq. (21) for the variable k we get that $k = k^0$ which is given by:

$$k^0 = \frac{-i(F^2 - 1) \pm \sqrt{(F^2 - 4)(F^2 - 1)}}{2(F^2 - 1)} \tag{22}$$

Replacing Eq. (22) in Eq. (16) ω^0 can be determined:

$$\omega^0 = \omega(k^0) = \frac{-i(F^4 + F^2 - 2) \pm i(F^2 + 1)\sqrt{(F^2 - 4)(F^2 - 1)}}{2F^2(F^2 - 1)} \tag{23}$$

that presents different results than those obtained by Di Cristo and Vacca (2005) (equation 3.8), as shown in Eq. (24):

$$\omega^0 = \frac{-i(F^2 - 2) \pm i\sqrt{(F^2 - 4)(F^2 - 1)}}{2F^2} \tag{24}$$

For the present work, when studying the original one developed by those authors, a reassessment was made in Eq. (16) to find what possible equation could have been employed to obtain the Eq. (24). In order to obtain the result presented by them, the temporal branches should have been defined as:

$$\omega_{\pm}(k) = k + \frac{i}{F^2} \pm \sqrt{\frac{k}{F^2}(k + i) - \frac{1}{F^4}}, \tag{25}$$

a solution that is not in agreement with the previous result found in Eq. (16).

Considering $\omega = \omega^0$ in Eq. (23), it is possible to conclude that the instability of the Saint-Venant flow model has a convective nature, since there are only finite singularities of the type pinch characterized by negative imaginary parts, and this occurs for values of $F > 2$, as shown in Fig. 4.

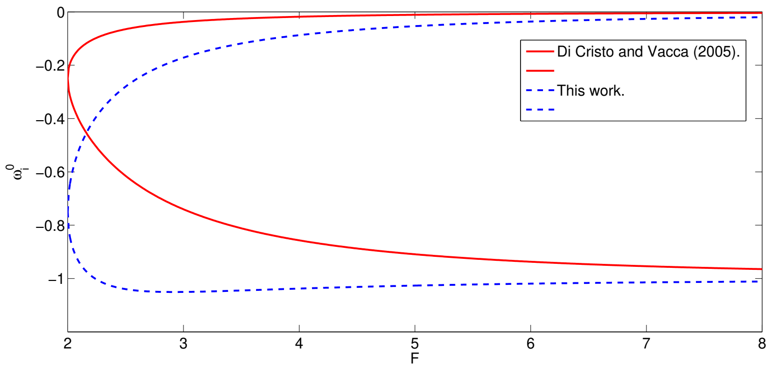

Figure 4. Imaginary part of finite singularities like pinch versus Froude number.

The Fig. 4 provides evidence of the negative definition of the ω_i^0 function. Thus, as the literature points out, there is a growth rate of instability ω_i for values of F greater than 2 with ω_i^0 always negative, featuring convective type instabilities. However, in this reassessment of the problem, the equation ω^0 presented by Di Cristo and Vacca (2005), and the one determined in this work, are different, as illustrated by Fig. 4.

4. CONCLUSIONS

In the present work, the Briggs criterion was applied, referring to the analysis of the singularities of the branching point of the dispersion relation. An analysis of linear spatial stability was performed that corroborated the convective nature of the instability.

The results showed that the rate of temporal growth of the unstable wave depends strongly on the frequency of the disturbance and the control parameter F, as predicted by the theory.

Based on the analysis of the wave propagation velocity, it was noticed that there is a strong dependency of the velocity on the wavenumber for small values of k. It has been noticed for large wavenumber the velocity propagation is constant. Based on the analysis carried through Fig. 3, it was possible to obtain the domain for possible wave propagation velocities $1 < U < \dfrac{3}{2}$, which is in agreement with analysis carried by Needham and Merkin (1984).

Finally, it was confirmed the work of Di Cristo and Vacca (2005) as an important part of the roll waves literature, even though a verified discrepancy on some features $\omega(k^0)$ were detected, where a sign in Eq. (25) made a difference in the geometric representation of finite singularities. In spite of this mathematical discrepancy, the analysis did not show any variation on the convective nature of the instability.

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